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OBSERVATIONS ON THE DIOPHANTINE EQUATION $x^2 + xy + y^2 = 12z^2$

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Abstract

A new and different set of solutions to the ternary quadratic equation $x^2 + xy + y^2 = 12z^2$ is obtained through the concept of geometric progression and Pythagorean equation.

Keywords: *homogeneous quadratic, ternary quadratic, integer solutions.*

I. INTRODUCTION

It is quite obvious that Diophantine equations are rich in variety [1,3] and occupy a remarkable position since antiquity. In particular, while searching for problems in quadratic diophantine equations, the paper [4] was noticed, wherein, the author have considered the ternary quadratic diophantine equation represented by $x^2 + xy + y^2 = 12z^2$ for non-zero distinct integer solutions and have presented some patterns of solutions. However, it is observed that there may be some more interesting sets of solutions to considered equation which is the motivation for our present communication. Four more new and different sets of solutions to the above equation are obtained through employing the concept of geometric progression and also the most cited solution of the Pythagorean equation. As far as our knowledge goes, it seems that the above solutions have not been presented earlier.

II. METHOD OF ANALYSIS

The ternary quadratic equation under consideration is

$$x^2 + xy + y^2 = 12z^2 \tag{1}$$

Introduction of the linear transformations

$$\left.\begin{aligned} x &= 2u + 6v \\ y &= 2u - 6v \end{aligned}\right\} \tag{2}$$

in (1) leads to

$$u^2 + 3v^2 = z^2 \tag{3}$$

Let a, b, c be three non-zero distinct integers.
Substituting

$$\left.\begin{aligned} v &= 2\alpha a \\ z &= b + 3\alpha^2 c \\ u &= b - 3\alpha^2 c \end{aligned}\right\}, \alpha \succ 0 \tag{4}$$

In (3), it simplifies to $a^2 = bc$ (5)
which implies that the triple (b, a, c) or (c, a, b) forms a G.P.

Note that (5) is satisfied by the following choices:

(i). $b=\sigma^2 c, a=\sigma c$

(ii). $c=\sigma^2 b, a=\sigma b$

Now, consider choice (i). From (4) and (2), the values of x, y, z satisfying (1) are given by

$$x=2c(\sigma^2-3\alpha^2)+12\alpha\sigma c$$
$$y=2c(\sigma^2-3\alpha^2)-12\alpha\sigma c$$
$$z=(\sigma^2+3\alpha^2)c$$

Observations:

1. Each of the following expressions is a Nasty number
   - $3c(x+y+4z)$
   - $c(4z-x-y)$
2. The triple $\left(x, y, 8c(\sigma^2-3\alpha^2)\right)$ is a diophantine three triple with the property $D(144\alpha^2\sigma^2c^2)$

For choice (ii), the solutions to (1) are obtained as :

$$x=2b(1-3\alpha^2\sigma^2)+12\alpha\sigma b$$
$$y=2b(1-3\alpha^2\sigma^2)-12\alpha\sigma b$$
$$z=b(1+3\alpha^2\sigma^2)$$

Observations:

1. Each of the following expressions is a Nasty number
   - $3b(x+y+4z)$
   - $b(4z-x-y)$
2. The triple $\left(x, y, 8b(1-3\alpha^2\sigma^2)\right)$ is a diophantine three triple with the property $D(144\alpha^2\sigma^2b^2)$

Further, taking

$$\left.\begin{aligned} b&=p+q \\ c&=p-q \end{aligned}\right\}, \; p\neq q\neq 0 \qquad (6)$$

in(5), it is written as

$$p^2=a^2+q^2$$

which is the well-known Pythagorean equation satisfied by the choices

(iii). $a=2rs, q=r^2-s^2, p=r^2+s^2, r\succ s\succ 0$

(iv). $a=r^2-s^2, q=2rs, p=r^2+s^2, r\succ s\succ 0$

Considering choice (iii), the values of x, y, z satisfying (1) are given by

$$x=4r^2-12\alpha^2s^2+24\alpha rs$$
$$y=4r^2-12\alpha^2s^2-24\alpha rs$$
$$z=2r^2+6\alpha^2s^2$$

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Observations:

1. Each of the following expressions is a Nasty number
    - $6(x+y+z)$
    - $2(4z-x-y)$
2. The triple $\left(x, y, 16r^2-48\alpha^2 s^2\right)$ is a diophantine three triple with the property $D(24^2\alpha^2 r^2 s^2)$

For choice (iv), the corresponding solutions to (1) are obtained as

$$x = 2(r+s)^2 - 6\alpha^2(r-s)^2 + 12\alpha(r^2 - s^2)$$

$$y = 2(r+s)^2 - 6\alpha^2(r-s)^2 - 12\alpha(r^2 - s^2)$$

$$z = (r+s)^2 + 3\alpha^2(r-s)^2$$

Observations:

1. Each of the following expressions is a Nasty number
    - $3(x+y+4z)$
    - $(4z-x-y)$
2. The triple $\left(x, y, 8(r+s)^2 - 24\alpha^2(r-s)^2\right)$ is a diophantine three triple with the property $D\left(144\alpha^2\left(r^2-s^2\right)^2\right)$

III. CONCLUSION

As the diophantine equations are rich in variety, one may attempt for obtaining integer solutions to other choices of quadratic equations with multiple variables.

REFERENCES

1. *Dickson. L.E., "History of the Theory of Numbers", Vol 2, Diophantine analysis, New York, Dover, 2005.*
2. *Mordell. L.J., "Diophantine Equations, Academic Press, New York, 1969.*
3. *Carmichael.R.D, "The Theory of numbers and Diophantine Analysis", New York, Dover, 1959.*
4. *R.Anbuselvi, S.A.Shanmugavadivu, "On The Non-Homogeneous Ternary Quadratic Equation $x^2 + xy + y^2 = 12z^2$ ", IOSR-JM, vol.12, Issue 1, Ver.III (Jan-Feb 2016), Pp.75-77.*